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SEMIALGEBRAIC EQUILIBRIUM MODELS FOR PLATE EXTENSION†

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Abstract—A finite strip equilibrium approach for the aproximate analysis of plane stress elastic problems is exposed and some stress models are developed. The related complementary energy minimization is turned toward a matrix-displacement procedure of solution, which is outlined for the case of a rectangular, in-plane-loaded, flat plate. Some numerical examples end the article.

1. INTRODUCTION

Semialgebraic, or semianalytical, approximate methods of analysis have been proposed for quite a while in the literature of structural mechanics.‡ The idea of merging algebraic and trigonometric functions to shape a discretized displacement field seems first due to Wilson[1], with regard to the analysis of axisymmetric solids. Later, a systematic, semialgebraic approach was introduced by Cheung[2] with the "finite strip method," devised primarily for plate-bending problems. Subsequently, such methods have been further developed by a number of authors with reference to slab-beam and box bridge structures, folded plate structures and vibration and stability problems, as shown in textbooks[3, 4] as well as in recent articles[5–10]. The most appreciable feature of these approaches is the marked computational convenience in comparison with (algebraic) finite element methods, besides an apparent versatility with respect to merely analytical methods. However, their applicability is confined to specific but important structural typologies.

Semialgebraic displacement models have been developed extensively, also with regard to two-dimensional elasticity problems. On this subject, in a recent paper by Cheung and Tham[11] a mixed model is proposed that leads to a least-squares formulation involving stresses and displacements. On the other hand, equilibrium approaches do not seem to be considered in the literature, except for a short communication given in [12]. However, an equilibrium approach allows one to look into a problem from a standpoint opposite to that of a compatible one (strain energy is bounded from above and from below, respectively), thus offering the benefit of a comparative analysis of results[13]. Moreover, such an approach provides for a stress field in equilibrium with the loads acting on a structure and permits, *coeteris paribus*, a better accuracy in assessing stresses at a point (besides continuity of tractions across an interdomain), and both these facts may be of a prominent interest for an engineer, if he needs to know displacements only at some relevant points of the structure at hand.

For the above reasons, a semialgebraic stress modeling for plate extension problems is presented in this paper. The exposition is taken with regard to a rectangular strip element (Sections 2–4), which is in principle suitable for describing a structural assembly of flat plates, but the approach can be extended, *mutatis mutandis*, to different geometries, e.g. curved strips. Indeed, the exposed models and the Southwell analogue[14] of some models due to Cheung[2] and Loo and Cusens[15, 16] for rectangular plates in bending, and subsequently extended to a number of cases in flexure. No restriction applies to possible end conditions for the strip. This feature could be advantageous, as displacement strip models in extension seem to have been developed

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‡ The author's preference is to use "semialgebraic" rather than the more popular "semianalytical" because the first adjective emphasizes the modern, and significant in practice, aspect of such methods.

for few cases of end conditions ([3], Chap. 3). The subsequent method of analysis takes place in the framework of the minimum complementary energy principle, and by this feature it could be connected to Papkovich's (e.g. [17], Sec. 83) or else Grioli's[18] methods, as well as Cheung relates its approach to Kantorovich's method[19]. As a consequence, a matrix-force procedure of solution would be a natural, but in some measure uneasy, development. However, in this paper, a matrix-displacement procedure is outlined, following the line due to Fraeijs De Veubeke[13], so that all the formal features of a displacement approach are preserved for a practical convenience. This aspect is drawn for the case of a rectangular, in-plane-loaded, flat plate (Section 5). Some comments and results of numerical investigations end the paper (Section 6).

2. BASIC RELATIONS

Reference is made to a flat, rectangular strip of depth $2b$ and length l, whose middle plane S is spanned by the Cartesian reference frame $(0, x, y)$ (Fig. 1). A point belonging to S is denoted by the vector $\mathbf{P}^T \equiv | x \; y |$ of its coordinates.† The strip is submitted to a plane state of stress that is referred to S. The components of generalized surface loads and displacements at $\mathbf{P}$ are collected in the vectors

$$\boldsymbol{\rho}^T \equiv | \rho_x \rho_y |, \qquad \mathbf{u}^T \equiv | u_x u_y |,$$

and the components of the (symmetric) generalized stress and strain tensors are ranged in the vectors

$$\mathbf{s}^T \equiv | s_x s_{xy} s_y |, \qquad \mathbf{e}^T \equiv | e_x e_{xy} e_y |,$$

respectively. Tractions and/or displacements along the edges $x = 0$, $x = l$ are supposed to be given in advance. No conditions are assigned along the edges $y = \pm b$.

Equilibrium equations in S are collected in the vector equation

$$\mathbf{D}^T\mathbf{s} + \boldsymbol{\rho} = \mathbf{0}, \tag{1}$$

where

$$\mathbf{D}^T \equiv \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & 0 \\ 0 & \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \end{vmatrix}$$

is the equilibrium matrix differential operator. Boundary tractions are ranged in the vector

$$\mathbf{t}^T \equiv | t_x t_y |$$

and related to stresses by the equation

$$\mathbf{t} = \mathbf{N}^T\mathbf{s},$$

where

$$\mathbf{N}^T(\mathbf{P}) \equiv \begin{vmatrix} n_x & n_y & 0 \\ 0 & n_x & n_y \end{vmatrix}$$

† Boldface letters denote vectors or matrices, superscript T means transpose. Subscript $,x$ $(,y)$ denotes partial derivative with respect to $x(y)$. A prime denotes derivative, unless otherwise declared.

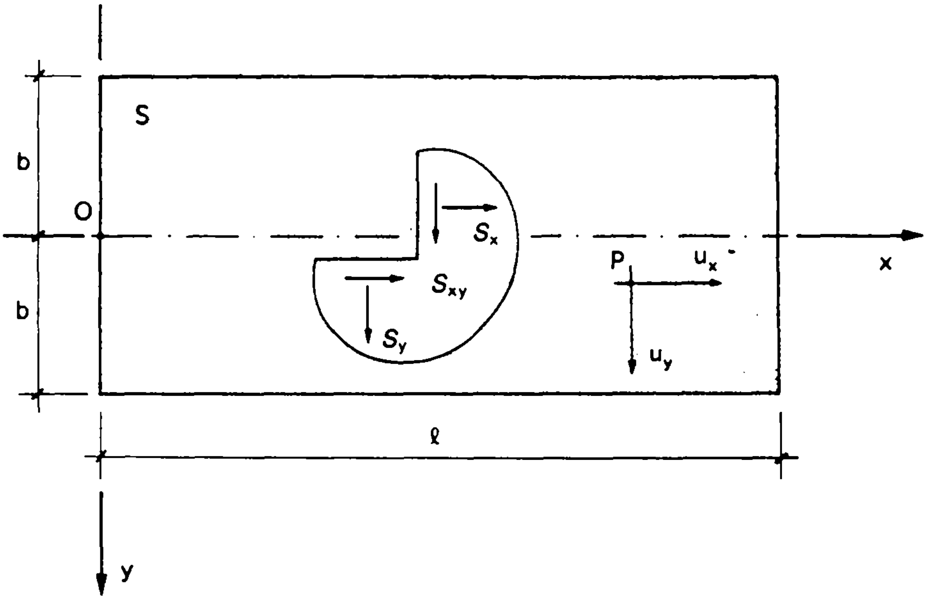

Fig. 1. Stresses and displacements in the strip.

and n_x, n_y are the direction cosines for the outward normal to the boundary at point **P**.

Stress field in S is represented by splitting vector s into two parts:

$$\mathbf{s} = \mathbf{q} + \mathbf{g}.$$

Vector **q** is a solution of eqn (1) with zero surface loads and homogeneous conditions along the vertical edges where tractions are prescribed. It incorporates stress fields in equilibrium with surface loads ρ and possibly prescribed tractions along edges $x = 0$, $x = l$. This provided, stress field **q** is kept indeterminate, while **g** is given.

Strains and stresses at **P** are related by means of the relationship

$$\mathbf{e} = \mathbf{Hs},$$

where **H** is the (symmetric, positive definite) generalized elastic compliance matrix, whose elements are given functions of **P** in S.

3. STRESS FUNCTION AND STRESS FIELDS

Vector **q** is preliminarily referred to a stress function $F(\mathbf{P})$ by letting

$$| q_x q_{xy} q_y |^T = | F_{xx} \; - F_{xy} \; F_{yy} |^T, \tag{2}$$

as customary in order to satisfy identically eqn (1) with null body forces. Stress function is represented in the factored form

$$F(\mathbf{P}) = \sum_j X_j(x) Y_j(y), \tag{3}$$

where functions $X_j(x)$ and $Y_j(y)$ are suitably assumed, and eqn (2) yields

$$q_x = \sum_j X_j Y_j'', \qquad q_{xy} = -\sum_j X_j' Y_j', \qquad q_y = \sum_j X_j'' Y_j. \tag{4a,b,c}$$

3.1 *Functions* $X_j(x)$

An infinite set of linearly independent functions $X_j(x)$ can be obtained from the eigenvalue problem defined by the homogeneous, linear, differential equation

$$X''''(\xi) + \alpha^2 X''(\xi) - \beta^4 X(\xi) = 0, \qquad 0 < \xi < 1, \quad \xi = \frac{x}{l}, \tag{5}$$

where either of the real parameters α and β is fixed in advance and possible conditions to be met at each end of the interval $0 \leq \xi \leq 1$ are

$$X = 0 \quad \text{or else} \quad X''' + \alpha^2 X' = 0 \tag{6a,b}$$

$$X' = 0 \quad \text{or else} \quad X'' = 0. \tag{6c,d}$$

The actual end conditions for eqn (5) should be selected to accommodate the prescribed conditions at the ends of the strip. Namely, if tractions t_x, t_y are prescribed along a vertical edge, then conditions

$$q_x = 0, \qquad q_{xy} = 0 \tag{7a,b}$$

must be obeyed on this edge, and taking into account eqn (4), conditions (6a) and (6c) apply to eqn (5) at the corresponding end of the interval. On the other hand, if displacements u_x, u_y are prescribed, then no restraint is imposed on q_x, q_{xy} and conditions (6b) and (6d) take place. As a consequence, four independent appropriate boundary conditions are attached to eqn (5) for a given situation at the ends of the strip, and the related sequences of eigenvalues and eigenfunctions are deduced by starting from the general solution of eqn (5):

$$X(\xi) = X(0)[B_1(\xi) + \alpha^2 B_2(\xi)] + X'(0)[B_3(\xi) + \alpha^2 B_4(\xi)] + X''(0)B_2(\xi) + X'''(0)B_4(\xi), \tag{8}$$

where

$$B_1(\xi) = (\epsilon^2 + \delta^2)^{-1}(\delta^2 \cosh \delta\xi + \epsilon^2 \cos \epsilon\xi)$$

$$B_2(\xi) = (\epsilon^2 + \delta^2)^{-1}(\cosh \delta\xi - \cos \epsilon\xi)$$

$$B_3(\xi) = (\epsilon^2 + \delta^2)^{-1}(\delta \sinh \delta\xi + \epsilon \sin \epsilon\xi)$$

$$B_4(\xi) = (\epsilon^2 + \delta^2)^{-1}(\delta^{-1} \sinh \delta\xi - \epsilon^{-1} \sin \epsilon\xi)$$

$$\delta = [-0.5\alpha^2 + (0.25\alpha^4 + \beta^4)^{1/2}]^{1/2}$$

$$\epsilon = [0.5\alpha^2 + (0.25\alpha^4 + \beta^4)^{1/2}]^{1/2}.$$

In this way, conditions (7a) and (7b) are fulfilled term by term in eqns (4a) and (4b), whereas conditions on displacements are taken into account by giving place to corresponding, indeterminate restraint forces.

It is worthwhile to recognize that eqns (5) and (6) are related to the problem of transverse vibrations of a straight, elastic beam of constant cross-section subjected to a constant axial force[20], where $X(\xi)$ is the deflection at a point of the centroidal axis, and the left-hand sides of eqns (6c) and (6d) are proportional to slope and bending moment and shearing force, respectively, in a cross-section. As a consequence, by setting $\beta = 0$ in eqn (5), the equation of the buckling problem of a column is obtained, and its general solution is deduced as the limit of the right-hand side of eqn (8) for $\beta \rightarrow 0$. On the other hand, the equation of the natural vibration problem of a beam and its general solution are obtained by setting $\alpha = 0$ in eqns (5) and (8), respectively. The eigenfunctions related to the latter problem have been directly employed for semialgebraic displacement strip models in bending[3]. In the present approach, however, static (force) end conditions for the strip are related to geometric (displacement) end conditions for the beam and vice versa. For the sake of completeness, the eigenfunctions related to some significant end conditions for the strip are reported with reference to $\beta = 0$ and $\alpha = 0$ alternatively prescribed in eqn (5).

(i) *Cantilever plate restrained at* x = 1. The eigenfunctions of the cantilever beam clamped at $x = 0$ are employed:

$$X_i = 1 - \cos \alpha_i x, \qquad \alpha_i l = \frac{2i - 1}{2}\pi, \qquad i = 1, 2, \ldots \tag{9a}$$

$$X_i = \sin \beta_i x - \sinh \beta_i x - \frac{\sin \beta_i l + \sinh \beta_i l}{\cos \beta_i l + \cosh \beta_i l}(\cos \beta_i x - \cosh \beta_i x), \tag{9b}$$

$$\beta_i l = 1.8751, 4.6941, 7.8548, 10.9956, \ldots, \frac{2i - 1}{2}\pi, \ldots, \qquad i = 1, 2, \ldots.$$

(ii) *Displacement* u_y *restrained, displacement* u_x *free along both ends of the strip.* The eigenfunctions are those of the beam hinged at the ends:

$$x_i = \sin \alpha_i x, \qquad \alpha_i l = i\pi, \qquad i = 1, 2, \ldots \tag{10}$$

For $\alpha = 0$ prescribed in eqn (5), relationship (10) holds with β_i replacing α_i. In the following (Section 6), the plate subjected to this case of constraint will be shortly named as simply supported, by analogy with a (deep) beam.

(iii) *Both ends free.* The eigenfunctions are those of the beam clamped at both ends:

$$x_i = 1 - \cos \alpha_i x, \qquad \alpha_i l = (i + 1)\pi, \qquad i = 1, 3, 5, \ldots \tag{11a}$$

$$x_i = \frac{2x}{l} - 1 - \frac{\sin \alpha_i(x - 0.5l)}{\sin \alpha_i(0.5l)} \tag{11b}$$

$$\alpha_i l = 2.8606\pi, 4.9180\pi, 6.9418\pi, 8.9548\pi, \ldots, (i + 1)\pi, \ldots, \qquad i = 2, 4, 6, \ldots$$

$$x_i = \sin \beta_i x - \sinh \beta_i x - \frac{\sin \beta_i l - \sinh \beta_i l}{\cos \beta_i l - \cosh \beta_i l}(\cos \beta_i x - \cosh \beta_i x),$$

$$\beta_i l = 4.7300, 7.8532, 10.9956, 14.1372, \ldots, \frac{2i + 1}{2}\pi, \ldots, \qquad i = 1, 2, \ldots \tag{11c}$$

(iv) *Displacements* u_x *restrained* ($q_x \neq 0$), u_y *free* ($q_{xy} = 0$) *along both ends of the strip.* The eigenfunctions are those of the guided beam, i.e. conditions (6b) and (6c) apply to eqn (5) at both ends of interval (0, l). For $\beta = 0$ prescribed in eqn (5), eigenvalue $\alpha = 0$ takes place, and the corresponding normalized eigenfunction

$$X_{01} = 1 \tag{12a}$$

is kept distinct for the sake of clearness. The subsequent eigenvalues have multiplicity two, and the related eigenfunctions are

$$X_i = \cos \alpha_i x, \qquad \alpha_i l = i\pi, \qquad i = 1, 2, \ldots \tag{12b}$$

besides the unit function, which is disregarded because it is already taken into account through X_{01}. Eigenfunctions (12) take also place for $\alpha = 0$ in eqn (5), with β_i replacing α_i.

(v) *Displacements* u_y *restrained* ($q_{xy} \neq 0$) *and* u_x *free* ($q_x = 0$) *along edge* $x = 0$, u_x *and* u_y *both restrained* ($q_x \neq 0$, $q_{xy} \neq 0$) *along edge* $x = l$. Conditions (6d) and (6a) at $x = 0$ and (6b) and (6d) at $x = l$ apply to eqn (5). For $\beta = 0$ prescribed, eigenvalue $\alpha = 0$ takes place, with the normalized eigenfunction

$$X_{02} = \frac{x}{l}. \tag{13a}$$

The subsequent eigenvalues are the same as those of the hinged beam and the corresponding eigenfunctions are given by (10). Eigenfunctions of eqn (5) with $\alpha = 0$ are (13a) and

$$X_i = \sin \beta_i x + \frac{\sin \beta_i l}{\sinh \beta_i l} \sinh \beta_i x, \tag{13b}$$

$$\beta_i l = 3.9266,\ 7.0661,\ 10.2102,\ 13.3520,\ \ldots,\ \frac{2i - 1}{4}\pi,\ \ldots, \qquad i = 1, 2$$

(vi) *Displacements* u_x, u_y *restrained* ($q_x \neq 0$, $q_{xy} \neq 0$) *along both ends of the strip*. Free end conditions apply to eqn (5). The null eigenvalue takes place with multiplicity two. The related eigenfunctions are (12a) and (13a). The subsequent eigenvalues of eqn (5) with $\beta = 0$ are again the same as those of the hinged beam, but their multiplicity is two. The corresponding eigenfunctions are those of eqn (10), besides the unit function, disregarded as in case (iv). Eigenfunctions of eqn (5) with $\alpha = 0$ are

$$X_i = \sin \beta_i x + \sinh \beta_i x - \frac{\sin \beta_i l - \sinh \beta_i l}{\cos \beta_i l - \cosh \beta_i l} (\cos \beta_i x + \cosh \beta_i x).$$

The eigenvalue sequence is the same as for X_i, eqn (11c).

Each sequence of eigenfunctions of eqn (5), for given boundary conditions, is a complete system of functions in $L^2(0, l)$. For $\alpha \geq 0$ fixed in advance, such functions are orthogonal (with the exclusion of eigenfunctions related to eigenvalue zero, if present), and their second derivatives are orthogonal as well if $\alpha = 0$ is prescribed. If $\beta = 0$ is assumed in eqn (5), orthogonality is lost for the relevant systems of eigenfunctions, except for cases (ii) and (iv), but the sets of their (nonnull) first and second derivatives are again orthogonal[21].

The above properties lead to employing eigenfunction systems related to $\alpha = 0$, or else $\beta = 0$, in eqn (5). However, the sets of their derivatives may be not complete, this fact depending on boundary conditions imposed, so that lack of completeness is possible for expansions (4b) and (4c), if based only on these sets. In this regard, let $f(x) = \bar{f} + \tilde{f}(x)$ be a function whose mean value on interval $(0, l)$ is $\bar{f}$. Integration by parts yields

$$\int_0^l f(x) X_k' \, dx = \bar{f}[X_k(l) - X_k(0)] + \tilde{f} X_k]_0^l - \int_0^l \tilde{f}' X_k \, dx, \tag{14}$$

index k ranging on the sequence of (nonnull) first derivatives of an eigenfunction system. Thus, it is apparent that a system X_k' comes to be complete only relatively to a function $f(x)$ whose mean value is zero, if functions X_k meet boundary conditions such that $X_k(0) = X_k(l)$ for each k. In such a case, eqn (4a) renders

$$q_x(0, y) = q_x(l, y), \tag{15}$$

so implying that end conditions $q_x(l, y) = q_x(0, y)$ are explicitly prescribed in advance, since boundary conditions are homogeneous and apply to each end of the strip independently of the other one. Moreover, this yields

$$\int_0^l q_{xy}\,\mathrm{d}x = -\sum_k \int_0^l X'_k Y'_k\,\mathrm{d}x = -\sum_k [X_k(l) - X_k(0)]\,Y'_k = 0,$$

so that eqn (4b) cannot represent a stress component q^c_{xy} constant with respect to x, $q^c_{xy} = h(y)$. On the other hand, eqn (15) and eqn (1) with $\rho_x = 0$ imply

$$h'(y) = -\frac{1}{l}\int_0^l q_{x,x}\,\mathrm{d}x = \frac{1}{l}[q_x(0, y) - q_x(l, y)] = 0.$$

Hence, the constant shear stress state

$$q^c_{xy} = a'_c \tag{16}$$

must be added, if admitted by the end conditions on the strip, to the right-hand side of eqn (4b), to complete the expansion of q_{xy}. This applies to case (ii).

Consider now the linear function $f(x) = (a_{c2} - a_{c1})x/l + a_{c1}$, a_{c1} and a_{c2} being constant, and the identity

$$\int_0^l f(x)X''_k\,\mathrm{d}x = \frac{a_{c2} - a_{c1}}{l}[lX'_k(l) - X_k(l) + X_k(0)] + a_{c1}[X'_k(l) - X'_k(0)], \tag{17}$$

where index k ranges on the system of (nonnull) second derivatives of an eigenfunction sequence. Suppose that the sum in the second square brackets is zero for each k, while the one in the first square brackets is nonzero for some k. Then, system X''_k cannot represent the constant function. On the other hand, this is equivalent, via eqn (4b), to

$$q_{xy}(0, y) = q_{xy}(l, y) \tag{18}$$

and matches conditions $q_{xy}(0, y) = q_{xy}(l, y) = 0$ prescribed in advance, as well as

$$\int_0^l q_y\,\mathrm{d}x = \sum_k \int_0^l X''_k Y_k\,\mathrm{d}x = \sum_k Y_k[X'_k(l) - X'_k(0)] = 0.$$

As a consequence, eqn (4c) cannot represent a stress component q_y constant with respect to x, $q^c_y = h(y)$. Moreover, eqn (18) and the second scalar equation of (1) with $\rho_y = 0$ yield

$$h'(y) = -\frac{1}{l}\int_0^l q_y\,\mathrm{d}x = \frac{1}{l}[q_{xy}(0, y) - q_{xy}(l, y)] = 0.$$

Hence, the constant stress state

$$q^c_y = a_{c1} \tag{19}$$

is the completion of expansion (4c), which applies to case (iv). If the sum in the first square brackets also vanishes—i.e. conditions $q_x(0, y) = q_y(l, y) = 0$ are prescribed too [case (iii)]—then expansion (4c) is unsuitable for representing the stress component

$$q_y = \left(\frac{a_{c2} - a_{c1}}{l}x + a_{c1}\right)h(y).$$

Moreover, on account of eqn (1) with $\boldsymbol{\rho} = 0$, function $h(y)$ must be linear. As a consequence, stress components

$$q_y^c = a_{c_1}\left(1 - \frac{x}{l}\right) + a_{c_2}\frac{x}{l} + \frac{6a_{c_3}}{l}\left(\frac{2x}{l} - 1\right)y \tag{20}$$

$$q_{xy}^c = a_{c_3}\frac{6x}{l}\left(1 - \frac{x}{l}\right) \tag{21}$$

are added to expansions (4c) and (4b), respectively, to complete them. Equations (20) and (21) identify parameters a_{c_1}, a_{c_2}, a_{c_3}:

$$a_{c_1} = q_y^c(0, 0), \qquad a_{c_2} = q_y^c(l, 0), \qquad a_{c_3} = \frac{1}{l}\int_0^l q_{xy}^c \, dx.$$

A more detailed analysis of eqn (17) could point out possible lacks of completeness for systems deriving from further, less significant end conditions for the strip. However, the above discussion is sufficient to show how a lack of completeness ever matches the possibility for the strip of undergoing rigid body motions because of a lack of geometric constraints along its vertical edges, since this meets with a vanishing of restraint forces along directions x and/or y. As a consequence, in such cases the non-complete expansion (4b) or (4c) represents a set of self-equilibrated tractions acting upon a horizontal section of the strip, whereas the relevant completion autonomously meets eqn (1) with $\boldsymbol{\rho} = 0$.

3.2 *Functions* $Y_j(y)$

$Y_j(y)$ functions are m-degree algebraic polynomials in the variable y. If Y_j multiplies an eigenfunction X_i, then Y_j, denoted by Y_i, is a complete interpolation polynomial,

$$Y_i = \mathbf{H}_m^T(y)\mathbf{C}_i, \qquad -b \leq y \leq b,$$

where $H_m(y)$ is a vector of shape functions, and C_i is a vector of indeterminate stress parameters,

$$\mathbf{C}_i^T \equiv |\, a_i a_i' a_i'' \ldots a_i^{(m-1)/2} b_i b_i' b_i'' \ldots b_i^{(m-1)/2} \,|$$

for m odd and

$$\mathbf{C}_i^T \equiv |\, a_i a_i' \ldots a_i^{(m-2)/2} b_i b_i' \ldots b_i^{(m-1)/2} d_i \,|$$

for m even. Parameters $a_i, a_i', \ldots, (b_i, b_i', \ldots)$ represent the values taken by Y_i, $Y_i', \ldots$ for $y = b(y = -b)$; d_i is the value taken by Y_i for $y = 0$. The relevant shape functions are widely reported in the literature (see, for example, [22]), and it seems needless to repeat them here. In particular, $Y_i(y)$ is an Hermite interpolation poynomial for m odd. Cubic and quintic polynomials are in practice of major interest, as shown later.

The factors of eigenfunctions X_{01} and X_{02}, denoted by Y_{01} and Y_{02}, respectively, contribute to stress components q_x and q_{xy} only, through the terms

$$q_{x_1}^0 = Y_{01}''(y) \tag{22a}$$

$$q_{x_2}^0 = \frac{x}{l}Y_{02}''(y), \qquad q_{xy}^0 = -\frac{1}{l}Y_{02}'(y). \tag{22b,c}$$

As a consequence, completeness is required only for the derivative Y_{01}'', Y_{02}', which are again interpolation polynomials.

If polynomial $Y_j(y)$ is cubic, then eqn (4) yields generalized stresses q_x, q_{xy} and

q_y, as linear, quadratic and cubic functions of y, respectively. Factors Y_{01} and Y_{02} are

$$Y_{01}(y) = \frac{y^2}{4}\left[\left(1 + \frac{y}{3b}\right) a''_{01} + \left(1 - \frac{y}{3b}\right) b''_{01}\right]$$

$$Y_{02}(y) = \frac{ly}{2b}\left[-\left(\frac{y}{2} + b\right) a'_{02} + \left(\frac{y}{2} - b\right) b'_{02} + \frac{2}{l}\left(\frac{y^2}{6} - \frac{b^2}{2}\right) d_{02}\right],$$

where a''_{01}, b''_{01}, a'_{02}, b'_{02} and d_{02} are indeterminate stress parameters. Equations (22) yield, respectively,

$$q^0_{x_1} = \frac{1}{2}\left[\left(1 + \frac{y}{b}\right) a''_{01} + \left(1 + \frac{y}{b}\right) b''_{01}\right] \tag{23a}$$

$$q^0_{x_2} = \frac{x}{2b}\left(-a'_{02} + b'_{02} + \frac{2y}{l} d_{02}\right) \tag{23b}$$

$$q^0_{xy} = -\frac{1}{2b}\left[-(y + b)a'_{02} + (y - b)b'_{02} + \frac{1}{l}(y^2 - b^2) d_{02}\right]. \tag{23c}$$

If polynomial Y_j is quintic, generalized stresses q_x, q_{xy} and q_y are third-, fourth-, and fifth-degree functions of y, respectively. Factors Y_{01} and Y_{02} take the form

$$\begin{aligned} Y_{01}(y) = \frac{y^2}{8}\bigg[&\left(-\frac{y^3}{10b^3} + \frac{y}{b} + 2\right) a''_{01} + b\left(\frac{y^3}{10b^3} + \frac{y^2}{6b^2} - \frac{y}{3b} - 1\right) a'''_{01} \\ &+ \left(\frac{y^3}{10b^3} - \frac{y}{b} + 2\right) b''_{01} + b\left(\frac{y^3}{10b^3} - \frac{y^2}{6b^2} - \frac{y}{3b} + 1\right) b'''_{01}\bigg] \\ Y_{02}(y) = y\bigg[&\left(\frac{y^4}{10b^4} + \frac{y^3}{16b^3} - \frac{y^2}{3b^2} - \frac{3y}{8b}\right) a'_{02} + \left(\frac{y^4}{20b^3} + \frac{y^3}{16b^2} - \frac{y^2}{12b} - \frac{y}{8}\right) a''_{02} \\ &+ \left(\frac{y^4}{10b^4} - \frac{y^3}{16b^3} - \frac{y^2}{3b^2} + \frac{3y}{8b}\right) b'_{02} + \left(\frac{y^4}{20b^3} - \frac{y^3}{16b^2} - \frac{y^2}{12b} + \frac{y}{8}\right) b''_{02} \\ &+ \left(\frac{y^4}{5b^4} - \frac{2y^2}{3b^2} + 1\right) d_{02}\bigg], \end{aligned}$$

where a''_{01}, a'''_{01}, b''_{01}, b'''_{01} and a'_{02}, a''_{02}, b'_{02}, b''_{02}, d_{02} are again indeterminate stress parameters, and the related expressions of $q^0_{x_1}$, $q^0_{x_2}$, q^0_{xy} can be similarly obtained from eqns (22).

On the ground of the above statements, eqns (4) can be rewritten in detail for cases (i)–(vi). Stress q_x takes the forms

$$q_x(x, y) = \sum_i X_i Y''_i, \qquad q_x(x, y) = q^0_{x_1} + q^0_{x_2} + \sum_i X_i Y''_i. \tag{24a,b}$$

Expression (24a) pertains to cases (i)–(iii). Expression (24b), taking into account eqns (22a,b), is related to case (vi) and to cases (iv) and (v) by dropping the terms $q^0_{x_2}$, $q^0_{x_1}$, respectively. For $m \geqslant 5$, eqns (24) yield

$$q_x(x, b) = \sum_i a''_i X_i, \qquad q_x(x, b) = a''_{01} + (a''_{02}x/l) + \sum_i a''_i X_i. \tag{25a,b}$$

Stress q_{xy} takes the forms

$$q_{xy}(x, y) = -\sum_i X'_i Y'_i \tag{26a}$$

$$q_{xy}(x, y) = q^c_{xy} - \sum_i X'_i Y'_i, \qquad q_{xy}(x, y) = q^0_{xy} - \sum_i X'_i Y'_i. \tag{26b,c}$$

Expression (26a) pertains to cases (i) and (iv). Expression (26b) pertains to cases (ii) and (iii); q^c_{xy} is given by eqns (16) and (21), respectively, and depends on a stress parameter only. Expression (26c) is related to cases (v) and (vi), with q^0_{xy} given by eqn (22c).

For $m \geq 3$, eqns (26) yield, respectively,

$$q_{xy}(x, b) = -\sum_i a'_i X'_i \tag{27a}$$

$$q_{xy}(x, b) = q^c_{xy} - \sum_i a'_i X'_i, \qquad q_{xy}(x, b) = a'_{02} - \sum_i a'_i X'_i. \tag{27b,c}$$

Finally, stress q_y is rewritten as

$$q_y(x, y) = \sum_i X''_i Y_i, \qquad q_y(x, y) = q^c_y + \sum_i X''_i Y_i \tag{28a,b}$$

for cases (i), (ii), (v) and (vi), and (iii) and (iv), respectively—see eqns (19) and (20) for q^c_y. Equation (28a) yields

$$q_y(x, b) = \sum_i a_i X''_i, \tag{29a}$$

and eqn (28b) yields

$$q_y(x, b) = a_{c1}\left(1 - \frac{x}{l}\right) + a_{c2}\frac{x}{l} + 6\frac{a_{c3}}{l}\left(\frac{2x}{l} - 1\right)\cdot b + \sum_i a_i X''_i \tag{29b}$$

$$q_y(x, b) = a_{c1} + \sum_i a_i X''_i \tag{29c}$$

for cases (iii) and (iv), respectively.

Equations (25), (27) and (29) can be referred to the upper side of the strip by replacing b with $-b$, and a''_i, a'_i, a_i with b''_i, b'_i, b_i, respectively.

4. IN-PLANE LOADS

Some loading modes of current interest are considered for the sake of completeness. With reference to cases (i) and (iv)–(vi), a particular integral of eqn (1) for a surface load $\rho_x(x, y)$ over the rectangle $x_A \leq x \leq x_B$, $y_A \leq y \leq y_B$, $0 \leq x_A, x_B \leq l$, $-b \leq y_A, y_B \leq b$ (patch load) is given by the stress field

$$\begin{aligned}
g_x &= \varphi_x(x_A, y) - \varphi_x(x, y) && x_A \leq x \leq x_B,\ y_A \leq y \leq y_B\\
g_x &= \varphi_x(x_A, y) - \varphi_x(x_B, y) && x_B \leq x \leq l,\ y_A \leq y \leq y_B\\
g_x &= 0 && \text{elsewhere}\\
g_y &= g_{xy} = 0 && \text{in } S
\end{aligned}$$

$$\varphi_x(x, y) = \int \rho_x(x, y)\,dx.$$

Likewise, the stress state

$$\begin{aligned}
g_y &= -\varphi_y(x, y_A) && x_A \leq x \leq x_B,\ -b \leq y \leq y_A\\
g_y &= -\varphi_y(x, y) && x_A \leq x \leq x_B,\ y_A \leq y \leq y_B\\
g_y &= -\varphi_y(x, y_B) && x_A \leq x \leq x_B,\ y_B \leq y \leq b\\
g_y &= 0 && \text{elsewhere}\\
g_x &= g_{xy} = 0 && \text{in } S
\end{aligned}$$

$$\varphi_y = \int \rho_y(x, y)\,dy$$

is a solution of eqn (1) for the patch load $\rho_y(x, y)$ in all cases (i)–(vi). A solution for the case of constant body force is obtainable from the above expressions.

For cases (i) and (iv)–(vi), a line load $p_x(y)$ on the segment $x = x_B, y_A \leq y \leq y_B$, $0 \leq x_B \leq l$ can be accounted for by the stress state $g_x = p_x$ in the rectangle $x_B \leq x \leq l$, $y_A \leq y \leq y_B$, $g_x = 0$ elsewhere and $g_y = g_{xy} = 0$ in S.

A line load $p_y(y)$ parabolically distributed along the segment $x = x_B$, $-b \leq y \leq b$, $0 \leq x_B \leq l$ can be accounted for in case (i) through the stress field

$$
\begin{aligned}
g_x &= \frac{x}{2b}[p_y(b) - p_y(-b)] + \frac{xy}{b^2}[p_y(-b) + p_y(b) - 2p_y(0)] \\
g_{xy} &= \frac{1}{2}\left[p_y(-b)\left(\frac{y}{b} - 1\right) - p_y(b)\left(1 + \frac{y}{b}\right)\right] \\
&\quad + \frac{1}{2}\left(1 - \frac{y^2}{b^2}\right)[p_y(-b) + p_y(b) - 2p_y(0)] \\
g_y &= 0
\end{aligned}
$$

for $x_B < x \leq l$, and $\mathbf{g} = 0$ for $0 \leq x < x_B$. The same loading mode can take place in the cases (ii), (v) and (vi) through the stress field

$$
\begin{aligned}
g_x &= \frac{\xi}{2bl}\left\{p_y(b) - p_y(-b) + \frac{2y}{b}[p_y(-b) + p_y(b) - 2p_y(0)]\right\} \\
g_{xy} &= \frac{\lambda}{2l}\left\{p_y(-b)\left(\frac{y}{b} - 1\right) - p_y(b)\left(\frac{y}{b} + 1\right)\right. \\
&\quad \left. + [p_y(-b) + p_y(b) - 2p_y(0)]\left(1 - \frac{y^2}{b^2}\right)\right\} \\
g_y &= 0,
\end{aligned}
$$

where $\xi = x(l - x_B)$ and $\lambda = l - x_B$ for $0 \leq x < x_B$, $\xi = x_B(l - x)$ and $\lambda = -x_B$ for $x_B < x \leq l$.

Finally, it should be noted that the expansions of the previous section are suitable, for their completeness, to represent (in the mean) any loading mode upon the horizontal edges, so that a surface load problem is reduced to an edge load problem for each case of end constraints.

5. MATRIX-FORCE AND MATRIX-DISPLACEMENT PROCEDURES OF SOLUTION

The foregoing exposition leads in principle to a set of semialgebraic equilibrium models, characterized by the degree m of the polynomial factors $Y_j(y)$ [eqn (3)]. Degree $m = 3$ is the lowest possible to shape a sufficiently comprehensive stress field in the strip. Indeed, in this case, all stress parameters are referred to independent equilibrium conditions along the horizontal edges, unless further, statically redundant stress parameters—appearing in the terms related to eigenvalue zero—take place because of the constraints at the ends the strip. Polynomial factors of degree $m > 3$ give inner redundancy to the stress field. An autonomous representation for stresses q_x along the horizontal sides of the strip becomes possible with a degree $m \geq 5$ for $Y_j(y)$ [see eqns (25)].

Each stress model gives rise to a deformation field that is not integrable (e.g. factor $Y_j(y)$ is a biharmonic function in Ribière's and Filon's solutions[17]). Nevertheless, displacements $\mathbf{u}_u(x)$, $\mathbf{u}_l(x)$ along upper and lower sides of the strip can be related to the assumed stress field by means of the complementary virtual work principle.

On this purpose, stress fields $\mathbf{q}$ and $\mathbf{g}$ are rewritten in the more concise form

$$
\mathbf{q} = \mathbf{Q}(\mathbf{P})\mathbf{r}, \qquad \mathbf{g} = \mathbf{G}(\mathbf{P})\mathbf{p},
$$

where vector $\mathbf{r}$ collects the set of indeterminate stress parameters, vector $\mathbf{p}$ collects the intensity factors of the applied loads and matrices $\mathbf{Q(P)}$ and $\mathbf{G(P)}$ range the relevant stress modes. Stress fields $\mathbf{q}$ and $\mathbf{g}$ produce, along the upper and lower sides of the strip, systems of tractions that can be expressed in the form

$$\mathbf{t}(\mathbf{P}_u) = \mathbf{T}(x)[\mathbf{R}_u\mathbf{r} + \mathbf{S}_u\mathbf{p}], \qquad \mathbf{P}_u^T \equiv |\, x \quad -b \,|$$

$$\mathbf{t}(\mathbf{P}_l) = \mathbf{T}(x)[\mathbf{R}_l\mathbf{r} + \mathbf{S}_l\mathbf{p}], \qquad \mathbf{P}_l^T \equiv |\, x \quad b \,|$$

with the following identifications for the lower side:

$$\mathbf{T}(x)\mathbf{R}_l \equiv \mathbf{N}^T(\mathbf{P}_l)\mathbf{Q}(\mathbf{P}_l), \qquad \mathbf{T}(x)\mathbf{S}_l \simeq \mathbf{N}^T(\mathbf{P}_l)\mathbf{G}(\mathbf{P}_l)$$

and analogous identifications for the upper one. Matrix $\mathbf{T}(x)$ collects the (mutually independent) boundary tractions modes onto upper and lower sides, and $\mathbf{R}_l$, $\mathbf{S}_l$, $\mathbf{R}_u$ and $\mathbf{S}_u$ are (independent of x) load connection matrices. $\mathbf{R}_l$ and $\mathbf{R}_u$ are, in fact, matrices such that vectors

$$\mathbf{r}_u = \mathbf{R}_u\mathbf{r}, \qquad \mathbf{r}_l = \mathbf{R}_l\mathbf{r}$$

play the role of indeterminate boundary traction amplitudes. Moreover, tractions due to stress field $\mathbf{g}$ onto each horizontal side are represented through the same system of basis functions as the tractions due to stress field $\mathbf{q}$ [sign $\simeq$ in the expression of $\mathbf{T}(x)\mathbf{S}_l$ above]. As a consequence, vectors

$$\mathbf{f}_u = \mathbf{R}_u\mathbf{r} + \mathbf{S}_u\mathbf{p}, \qquad \mathbf{f}_l = \mathbf{R}_l\mathbf{r} + \mathbf{S}_l\mathbf{p} \tag{30a,b}$$

play the role of generalized loads for the strip.

For the sake of generality, the straining effect of a temperature gradient is also considered, besides strains due to stress field $\mathbf{s}$. Then, strain field in the strip takes the form

$$\mathbf{e} = \mathbf{H}[\mathbf{Q(P)r} + \mathbf{G(P)p}] + \mathbf{\Theta}(\mathbf{P})\boldsymbol{\vartheta},$$

where $\mathbf{\Theta}$ is the matrix of thermal strain modes and vector $\boldsymbol{\vartheta}$ collects the relevant intensity factors.

The virtual stress field

$$\hat{\mathbf{s}} = \mathbf{Q(P)}\hat{\mathbf{r}}$$

is assumed, vector $\hat{\mathbf{r}}$ containing an arbitrary set of stress parameters. Related boundary tractions on the horizontal side are

$$\hat{\mathbf{t}}(\mathbf{P}_u) = \mathbf{T}(x)\mathbf{R}_u\hat{\mathbf{r}}, \qquad \hat{\mathbf{t}}(\mathbf{P}_l) = \mathbf{T}(x)\mathbf{R}_l\hat{\mathbf{r}}.$$

Inner virtual work reads

$$\int_S \hat{\mathbf{s}}^T\mathbf{e}\,\mathrm{d}S = \hat{\mathbf{r}}^T(\mathbf{Fr} + \mathbf{Lp} + \mathbf{W}\boldsymbol{\vartheta}),$$

where

$$\mathbf{F} = \int_S \mathbf{Q}^T\mathbf{HQ}\,\mathrm{d}S, \qquad \mathbf{L} = \int_S \mathbf{Q}^T\mathbf{HG}\,\mathrm{d}S, \qquad \mathbf{W} = \int_S \mathbf{Q}^T\mathbf{\Theta}\,\mathrm{d}S.$$

For the sake of simplicity in the exposition, nonhomogeneous conditions on displacements along the vertical edges of the strip are not considered. Hence, boundary virtual work takes the form

$$\int_0^l [\hat{\mathbf{t}}^T(\mathbf{P}_u)\mathbf{u}_u(x) + \hat{\mathbf{t}}^T(\mathbf{P}_l)\mathbf{u}_l(x)]\,\mathrm{d}x = \hat{\mathbf{r}}^T(\mathbf{R}_u^T\mathbf{u}_u^* + \mathbf{R}_l^T\mathbf{u}_l^*),$$

where vectors

$$\mathbf{u}_u^* = \int_0^l \mathbf{T}^T(x)\mathbf{u}_u(x)\,\mathrm{d}x, \qquad \mathbf{u}_l^* = \int_0^l \mathbf{T}^T(x)\mathbf{u}_l(x)\,\mathrm{d}x \tag{31a,b}$$

can be regarded as generalized boundary displacements. In fact, the previous statements show that $\mathbf{T}(x)$ takes the role of matrix of displacement modes, and the corresponding amplitudes for given displacements $\mathbf{u}_u$, $\mathbf{u}_l$ are obtained premultiplying vectors $\mathbf{u}_u^*$, $\mathbf{u}_l^*$ by the inverse of the relevant Gramian matrix[23]. Identifying inner and boundary works for any $\hat{\mathbf{r}}$ leads to the equation

$$\mathbf{R}_u^T\mathbf{u}_u^* + \mathbf{R}_l^T\mathbf{u}_l^* = \mathbf{F}\mathbf{r} + \mathbf{L}\mathbf{p} + \mathbf{W}\boldsymbol{\vartheta}, \tag{32}$$

which is the stress-displacement relationship connected with the assumed stress model. Matrix $\mathbf{F}$ is recognizable as the (symmetric, positive definite) flexibility matrix of the strip, as well as $\mathbf{L}$ and $\mathbf{W}$ can be viewed as matrices of influence coefficients. From a slightly different standpoint, eqn (32) is the stationary (minimum) condition of the complementary energy of the strip undergoing arbitrary displacements along the horizontal sides, besides given external loads and thermal strains.

Equation (32) can be solved for vector $\mathbf{r}$ and combined with eqns (30) to obtain the force-displacement relationship

$$\mathbf{f} + (\mathbf{R}\mathbf{F}^{-1}\mathbf{L} - \mathbf{S})\mathbf{p} + \mathbf{R}\mathbf{F}^{-1}\mathbf{W}\boldsymbol{\vartheta} = \mathbf{K}\mathbf{u}^*, \tag{33}$$

where

$$\mathbf{f}^T = |\,\mathbf{f}_u^T\mathbf{f}_l^T\,|, \qquad \mathbf{u}^{*T} = |\,\mathbf{u}_u^{*T}\mathbf{u}_l^{*T}\,|$$

$$\mathbf{R}^T = |\,\mathbf{R}_u^T\mathbf{R}_l^T\,|, \qquad \mathbf{S}^T = |\,\mathbf{S}_u^T\mathbf{S}_l^T\,|.$$

$$\mathbf{K} = \mathbf{R}\mathbf{F}^{-1}\mathbf{R}^T.$$

Matrix $\mathbf{K}$ is recognizable as the stiffness matrix of the strip with respect to generalized displacements $\mathbf{u}^*$, and the second and third terms on the left-hand side are generalized, equivalent boundary loads due to applied loads and temperature rise onto S.

It is worth noting that a completion term [eqns (16), (19) and (21)], which is connected to specific rigid body degrees of freedom, produces boundary tractions along the upper and lower sides of a strip, which are related to the same set of stress parameters, whereas boundary tractions produced by the other terms of an expansion are related to distinct stress parameters. As a consequence, matrix $\mathbf{R}$ contains a couple of linearly dependent rows per each possible rigid body degree of freedom, if admitted because of the constraints at the ends of the strip. Otherwise, $\mathbf{R}$ is a full row rank matrix. Hence, stiffness matrix $\mathbf{K}$ is singular only in the first evenience (the rank differing from the order for the number of possible rigid body degrees of freedom); i.e. the displacement model is free of any kinematical deformation modes[13]. For the same reason, only the elements of $\mathbf{u}_l^*$, $\mathbf{u}_l^*$ corresponding to the same rigid body displacement mode, if present, can appear simultaneously in an equation of relationship (32). Such an equation, in fact, involves the amplitude of a relative displacement mode, while any other equation involves an element of $\mathbf{u}_u^*$, or else $\mathbf{u}_l^*$, only.

In the framework of a structural modeling, all the strips of an assembly must have the same length and the same constraint conditions at the ends. As a consequence, stress field description in the local reference frame (0, x, y), i.e. matrices $\mathbf{Q}(\mathbf{P})$ and $\mathbf{T}(x)$, is formally the same everywhere in the assembly. Moreover, given loads (imposed displacements) in the x and y directions onto interstrip or boundary sides are represented through the amplitudes of their expansions in the systems of basis functions underlying the representation of q_{xy} and q_y, respectively. Continuity of stresses across common boundaries would be imposed in principle, taking into account the interstrip loads. However, the essential requirement of a discrete equilibrium model is simply to meet continuity of mutual tractions among contiguous strips. Therefore, elements of vector $\mathbf{r}$, not related with $\mathbf{r}_u$, $\mathbf{r}_l$, if they are present, can be considered only at the strip level, without including them at the level of assembly. Taking into account the role of matrices $\mathbf{R}_u$, $\mathbf{R}_l$, it becomes apparent that as many equations of relationship (32) have null left-hand sides (i.e. are not involved with boundary displacements, null columns of $\mathbf{R}_u$, $\mathbf{R}_l$) as stress parameters not related with $\mathbf{r}_u$, $\mathbf{r}_l$ are contained in vector $\mathbf{r}$. As a consequence, those equations can be solved for these stress parameters, which are subsequently eliminated from the remaining equations (condensation). In such a way, continuity on stress s_x across contiguous strips is not enforced. On the other hand, if continuity of stress s_x is required too—provided that polynomial factors $Y_j(y)$ are taken with degree $m = 5$ at least—then stress g_x along a common boundary is possibly needed to be represented in the system of eigenfunctions $X_j(x)$, to enter the relevant amplitudes into the set of continuity conditions in terms of generalized parameters. A possible condensation of relationship (32) is now performed only on those stress parameters that do not appear in the expressions of q_x, q_{xy}, q_y for $y = \pm b$. The condensed stress-displacement relationship replaces eqn (32) in every respect.

Equation (32) is the basis of a matrix-force procedure of analysis. Once boundary equilibrium and stress continuity conditions have been imposed, a set of statically indeterminate stress parameters rules the stress field in the whole assembly. Such parameters are then determined by enforcing compatibility along common or constrained strip sides. For this purpose, the complementary virtual work principle can again be invoked. For instance, compatibility between two contiguous, coplanar strips j and k in terms of complementary virtual work reads

$$\int_0^l [\mathbf{T}(x)\mathbf{R}_{lj}\hat{\mathbf{r}}]^T \mathbf{u}_{lj}(x)\,dx + \int_0^l [\mathbf{T}(x)\mathbf{R}_{uk}\hat{\mathbf{r}}]^T \mathbf{u}_{uk}(x)\,dx = 0$$

for any $\hat{\mathbf{r}}$, $\mathbf{u}_{lj}(x)$ and $\mathbf{u}_{uk}(x)$ being distinct displacement distributions along the sides of (disconnected) strips j and k where compatibility is enforced. The above equation yields, via statements (31), the required compatibility condition in terms of generalized displacements,

$$\mathbf{R}_{lj}^T\mathbf{u}_{lj}^* + \mathbf{R}_{uk}^T\mathbf{u}_{uk}^* = \mathbf{0},$$

and a glance at eqn (32) shows that the overall flexibility matrix as well as the vector of generalized known displacements for a given structural assembly could be readily obtained by overlapping from the analogous, individual arrays of the strips, when stress parameters due to possibly present completion terms of the stress field in the typical strip are statically determined. Otherwise, the assembling of the system of compatibility equations could become in some manner complicated, owing to the presence of equations involving flexibilities of noncontiguous strips. Once all stress parameters are determined, generalized displacements at a strip side flow from eqn (32) and stresses and displacements at a point can be obtained through the relevant expansions. In practice, distributed (e.g. body) loading modes and interstrip loading limit the straightforwardness of the procedure.

A quite conventional matrix-displacement procedure of solution is supplied by eqn (33) and is now sketched for the case of an in-plane-loaded, flat, rectangular plate. As customary, the disconnected strips are assembled by identifying the individual, generalized displacements between common boundaries, to meet *a priori* interstrip compatibility. This can be represented formally, by letting for the typical sth strip

$$\mathbf{u}_s^* = \mathbf{B}_s \mathbf{w}^*, \tag{34}$$

where $\mathbf{u}_s^*$ is the vector of generalized side displacements of the disconnected strip, $\mathbf{w}^*$ is the vector ranging the set of generalized interstrip and boundary displacements through the assembled plate, and $\mathbf{B}_s$ is the appropriate localizing (Boolean) matrix. External loads acting on interstrip or boundary lines are entered by the vector of amplitudes $\boldsymbol{\varphi}$. The sum of virtual works performed by the external loads on the disconnected strips is set equal to the total virtual work of line loads acting on the assembled structure by means of the equation

$$\sum_s \mathbf{f}_s^T \mathbf{u}_s^* = \boldsymbol{\varphi}^T w^*, \tag{35}$$

where $\mathbf{f}_s$ is the vector of boundary generalized loads for the typical strip. Then, eqn (34) is substituted into (35) and identification of the coefficients of $\mathbf{w}^*$ yields

$$\boldsymbol{\varphi} = \sum_s \mathbf{B}_s^T \mathbf{f}_s .$$

Substitution of eqn (33), written for the sth strip, leads to the equilibrium relationship for the assembly,

$$\boldsymbol{\varphi} + \sum_s \mathbf{B}_s^T(\mathbf{R}_s\mathbf{F}_s^{-1}\mathbf{L}_s - \mathbf{S}_s)\mathbf{p}_s + \sum_s \mathbf{B}_s^T\mathbf{R}_s\mathbf{F}_s^{-1}\mathbf{W}_s\boldsymbol{\vartheta}_s = (\sum_s \mathbf{B}_s^T\mathbf{K}_s\mathbf{B}_s)\mathbf{w}^*, \tag{36}$$

where summation at the right-hand side produces the overall stiffness matrix of the plate with reference to the generalized interstrip displacements. Once vector $\mathbf{w}^*$ is determined, stress amplitudes are obtained strip by strip by solving eqn (32) for the relevant vector $\mathbf{r}$.

In this procedure, the constraints of continuity on tractions between contiguous strips are removed in principle by Lagrangian multipliers, which are the interstrip displacements. Because displacements are interpolated on the same basis of boundary tractions, continuity of tractions is, in fact, imposed mode by mode, i.e. pointwise along an interstrip. In this way, the polynomial parts $Y_j(y)$ of the stress function and their first derivatives for adjacent strips are continuous (or undergo a given jump due to line loads, if present) across an interstrip line. Hence, this procedure also suggests the logical scheme for enforcing given transitional conditions on higher partial derivatives of individual stress functions with respect to y, provided that polynomials $Y_j(y)$ are of degree $m = 5$ at least. In this regard, enforcing continuity on stress s_x could be of prominent interest. Stresses s_x along the upper and lower sides of the strip are represented in the same form as tractions $\mathbf{t}$:

$$s_x(\mathbf{P}_u) = \mathbf{Z}(x)[\boldsymbol{\Psi}_u\mathbf{r} + \boldsymbol{\Xi}_u\mathbf{p}]$$

$$s_x(\mathbf{P}_l) = \mathbf{Z}(x)[\boldsymbol{\Psi}_l\mathbf{r} + \boldsymbol{\Xi}_l\mathbf{p}]$$

with the identifications

$$\mathbf{Z}(x)\boldsymbol{\Psi}_l\mathbf{r} \equiv q_x(\mathbf{P}_l), \qquad \mathbf{Z}(x)\boldsymbol{\Xi}_l\mathbf{p} \simeq g_x(\mathbf{P}_l)$$

for the lower side and analogous identifications for the upper one. Vectors $\boldsymbol{\epsilon}_u^*$ and $\boldsymbol{\epsilon}_l^*$ are introduced with the same role of vectors $\mathbf{u}_u^*$ and $\mathbf{u}_l^*$, so that eqn (32) is rewritten in the following form:

$$\boldsymbol{\Psi}_u^T\boldsymbol{\epsilon}_u^* + \mathbf{R}_u^T\mathbf{u}_u^* + \boldsymbol{\Psi}_l^T\boldsymbol{\epsilon}_l^* + \mathbf{R}_l^T\mathbf{u}_l^* = \mathbf{F}\mathbf{r} + \mathbf{L}\mathbf{p} + \mathbf{W}\boldsymbol{\vartheta},$$

and is solved for vector $\mathbf{r}$ to obtain

$$\mathbf{r} = \mathbf{F}^{-1}\overline{\mathbf{R}}^T\overline{\mathbf{u}}^* - \mathbf{F}^{-1}\mathbf{L}\mathbf{p} - \mathbf{F}^{-1}\mathbf{W}\boldsymbol{\vartheta}, \tag{37}$$

where

$$\overline{\mathbf{R}}^T \equiv |\, \boldsymbol{\Psi}_u^T \,\vdots\, \mathbf{R}_u^T \,\vdots\, \boldsymbol{\Psi}_l^T \,\vdots\, \mathbf{R}_l^T \,|, \qquad \overline{\mathbf{u}}^{*T} \equiv |\, \boldsymbol{\epsilon}_u^{*T}\mathbf{u}_u^{*T}\boldsymbol{\epsilon}_l^{*T}\mathbf{u}_l^{*T} \,|.$$

The amplitudes of stress modes ranged in matrix $\boldsymbol{\mathfrak{Z}}(x)$ are collected in the vectors

$$\boldsymbol{\sigma}_u = \boldsymbol{\Psi}_u\mathbf{r} + \boldsymbol{\Xi}_u\mathbf{p}, \qquad \boldsymbol{\sigma}_l = \boldsymbol{\Psi}_l\mathbf{r} + \boldsymbol{\Xi}_l\mathbf{p}$$

for the upper and lower sides, respectively. As a consequence, relationship (33) is replaced by

$$\bar{\mathbf{f}} + (\overline{\mathbf{R}}\mathbf{F}^{-1}\mathbf{L} - \overline{\mathbf{S}})\mathbf{p} + \overline{\mathbf{R}}\mathbf{F}^{-1}\mathbf{W}\boldsymbol{\vartheta} = \overline{\mathbf{K}}\overline{\mathbf{u}}^* \tag{38}$$

with the following identifications:

$$\bar{\mathbf{f}}^T \equiv |\, \boldsymbol{\sigma}_u^T\mathbf{F}_u^T\boldsymbol{\sigma}_l^T\mathbf{f}_l^T \,|, \qquad \overline{\mathbf{S}}^T \equiv |\, \boldsymbol{\Xi}_u^T \,\vdots\, \mathbf{S}_u^T \,\vdots\, \boldsymbol{\Xi}_l^T \,\vdots\, \mathbf{S}_l^T \,|$$
$$\overline{\mathbf{K}} = \overline{\mathbf{R}}\mathbf{F}^{-1}\overline{\mathbf{R}}^T.$$

Matrix $\overline{\mathbf{R}}$ has the same features as matrix $\mathbf{R}$; i.e. the presence of linearly dependent rows is due only to possible rigid body degrees of freedom, as stress $q_x(\mathbf{P}_l)$ is independent of stress $q_x(\mathbf{P}_u)$. Hence, the actual stiffness matrix $\overline{\mathbf{K}}$ has the same properties of matrix $\mathbf{K}$. The assembling of relationships (38) for disconnected strips into the overall load-displacement system for the plate follows the same formal path as before (provided that vector $\boldsymbol{\Psi}_u$ for each strip is previously mutiplied by -1), and leads to the equation

$$\mathbf{A}\boldsymbol{\varphi} + \sum_s \overline{\mathbf{B}}_s^T(\overline{\mathbf{R}}_s\mathbf{F}_s^{-1}\mathbf{L}_s - \overline{\mathbf{S}}_s)\mathbf{p}_s + \sum_s \overline{\mathbf{B}}_s^T\overline{\mathbf{R}}_s\mathbf{F}_s^{-1}\mathbf{W}_s\boldsymbol{\vartheta}_s = \left(\sum_s \overline{\mathbf{B}}_s^T\overline{\mathbf{K}}_s\overline{\mathbf{B}}_s\right)\overline{\mathbf{w}}^*,$$

where vector $\overline{\mathbf{w}}^*$ collects parameters $\boldsymbol{\epsilon}^*$ and $\mathbf{u}^*$ attached to each interstrip and boundary line, $\overline{\mathbf{B}}_s$ is the localizing matrix that refers vector $\overline{\mathbf{u}}_s^*$ of the sth strip to vector $\overline{\mathbf{w}}^*$ and $\mathbf{A}$ is a Boolean matrix that locates the elements of vector $\boldsymbol{\varphi}$ in the relevant equilibrium equations.

Indeterminate parameters collected in vector $\boldsymbol{\epsilon}^*$ may be regarded as (sign-reversed) amplitudes of dislocation modes conjugated with stress q_x. It should be noted that continuity of stress s_x at an interstrip, in the complementary energy approach, would flow simply from continuity of strains (natural condition), if met in solution, and the relevant Lagrangian multipliers in $\boldsymbol{\epsilon}^*$ would result in zero in this case. On the other hand, vector $\boldsymbol{\epsilon}_u^*(\boldsymbol{\epsilon}_l^*)$ for the upper (lower) strip of the assembly must be set equal to zero; otherwise stress q_x on the upper (lower) side of the plate vanishes. This condition applies independently of further possible constraints on displacements along the horizontal sides of the plate.

6. NUMERICAL EXAMPLES AND CONCLUDING REMARKS

An isotropic square plate of constant thickness, elastic modulus E and Poisson ratio $\nu = 0.3$, restrained along the vertical edges, free along the lower edge and submitted to a uniform load of components $t_x = 0$, $t_y = \bar{t}_y$ at the top edge is considered.

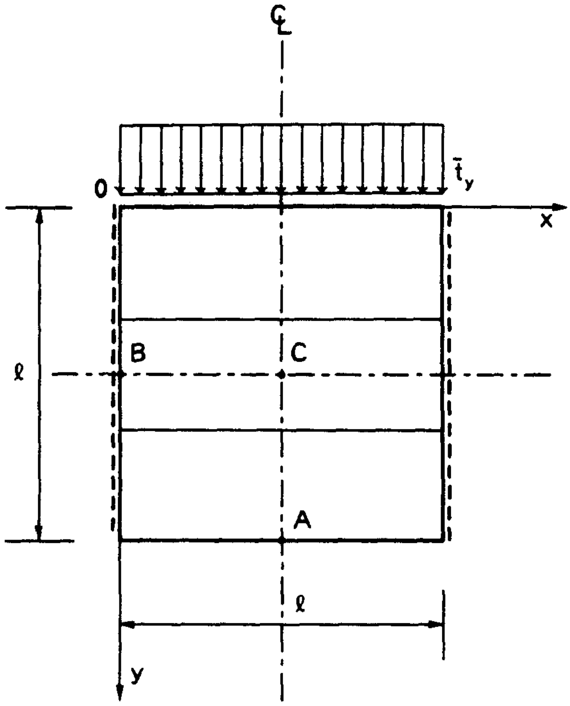

Fig. 2. Square plate simply supported along the vertical edges. NS = 3.

The plate is subdivided into equal horizontal strips. Models with functions $Y_j(y)$ respectively cubic (ES3) and quintic with continuity on stress s_x either relaxed (ES5) or enforced (HS5) are employed [in the code E stands for equilibrium, H for hyperequilibrium, S for strip and the numeral is in the degree m of $Y_j(y)$].† Flexibility matrices for models ES3 and HS5 (or ES5 before condensation) can be readily obtained by analogy with the stiffness matrices, in bending, of the lower-order rectangular strip, [2, p. 31] and the strip with curvature compatibility[24], so that it seems superfluous to report them.

Case (ii) (Section 3.1)—vertical edges simply supported—is considered first (Fig. 2), as a series solution due to Filon is available for it. Values taken by displacement u_y and stress s_x at point $A \equiv (l/2, l)$, stresses s_{xy} at point $B \equiv (0, l/2)$ and s_y at point $C \equiv (l/2, l/2)$, are shown in Tables 1–4 versus the number of strips (NS) and the number of (odd index) harmonics (NH) employed in the analysis. The comparison includes also the higher-order displacement strip HO3 (algebraic parts of u_x and u_y parabolic, three generalized coordinates per harmonic, one condensed out) proposed by Loo and Cusens[16, Sect. 3.3.2]. A superimposed line on a value means constant with respect to subsequent values of NH, up to 10 at least for the strip models, up to 50 for Filon's solution. The agreement of the equilibrium models with Filon's solution is apparent also for a coarse strip subdivision. Moreover, coincidence of results is soon obtained with models ES5 and HS5. This fact is by no means unexpected, because function (10) coincides with the trigonometric part of Filon's solution, as well as functions (12) coincide with Ribière's one, and a fifth-degree polynomial can fairly represent the biharmonic part of Filon's solution onto a sufficiently narrow interval, as recognized in [25, Art. 24]. Model HS5 appears to be slightly stiff in comparison with model ES5, whereas the more simple model ES3 would require a more refined subdivision for giving results closer to Filon's solution. It should be noted that functions (10) and (12) are orthogonal to their second derivatives. Thus, the contribution to the flexibility matrix for each index of the expansion decouples from the others, which is a nice feature from a computational standpoint.

The second case of constraint, a cantilever plate restrained at the right-hand edge

† Numerical developments were performed on the CDC Cyber 76 computer of CINECA, Bologna, Italy.

Table 1. Simply supported square plate uniformly loaded at the upper side (Fig. 2). Displacement u_x at point A (multiplier: $0.1 \cdot \bar{t}_x l/E$)

Models	NH	NS 1	NS 3	NS 5	Filon
ES3	1	3.0277	3.1622	3.1502	3.1481
	2	3.4522	3.1599	3.1486	3.1465
	3	3.1501	3.1595	3.1486	3.1465
ES5	1	3.1464	3.1481	3.1481	3.1481
	2	3.1862	3.1465	3.1465	3.1465
	3	3.1431	3.1465	3.1465	3.1465
HS5	1	3.1464	3.1481	3.1481	3.1481
	2	3.1862	3.1466	3.1465	3.1465
	3	3.1431	3.1465	3.1465	3.1465
HO3	1	2.2935	3.1232	3.1444	3.1481
	2	2.2772	3.1216	3.1429	3.1465
	3	2.3075	3.1215	3.1429	3.1465

Table 2. Simply supported square plate uniformly loaded at the upper side (Fig. 2.). Stress s_x at point A (multiplier: $0.1 \cdot \bar{t}_x$)

Models	NH	NS 1	NS 3	NS 5	NS 7	NS 9	Filon
ES3	1	7.7404	6.7106	7.1288	7.2828	7.3544	7.4807
	2	7.4537	6.7053	7.1197	7.2722	7.3431	7.4678
	3	7.5156	6.7097	7.1197	7.2722	7.3431	7.4678
ES5	1	7.7619	7.4725	7.4793	7.4803	7.4807	7.4807
	2	7.2130	7.4603	7.4665	7.4675	7.4678	7.4678
	3	7.4216	7.4603	7.4666	7.4675	7.4678	7.4678
HS5	1	7.7619	7.4714	7.4792	7.4803	7.4807	7.4807
	2	7.2130	7.4560	7.4664	7.4674	7.4678	7.4678
	3	7.4216	7.4626	7.4665	7.4675	7.4678	7.4678
HO3	1	4.1857	7.5794	7.5552	7.6214	7.5115	7.4807
	2	4.8984	7.5660	7.5417	7.6075	7.4982	7.4678
	3	4.6880	7.5647	7.5417	7.6075	7.4983	7.4678

Table 3. Simply supported square plate uniformly loaded at the upper side (Fig. 2). Stress s_{xy} at point B (multiplier: $0.1 \cdot \bar{t}_x$)

Models	NH	NS 1	NS 3	NS 5	NS 7	NS 9	Filon
ES3	1	6.0793	5.4806	5.4756	5.4749	5.4747	5.4746
	2	6.7547	5.6615	5.6579	5.6557	5.6550	5.6546
	3	6.9979	5.6904	5.6669	5.6640	5.6630	5.6623
	4	7.1220	5.7235	5.6680	5.6644	5.6634	5.6626
	5	7.1970	5.7556	5.6696	5.6645	5.6634	5.6627
ES5	1	5.4569	5.4746	5.4746	5.4746	5.4746	5.4746
	2	5.5349	5.6544	5.6545	5.6545	5.6546	5.6546
	3	5.4413	5.6620	5.6623	5.6623	5.6623	5.6623
	4	5.3689	5.6621	5.6627	5.6627	5.6627	5.6626
	5	5.3179	5.6616	5.6627	5.6627	5.6627	5.6627
HS5	1	5.4569	5.4748	5.4746	5.4746	5.4746	5.4746
	2	5.5349	5.6573	5.6545	5.6546	5.6546	5.6546
	3	5.4413	5.6716	5.6621	5.6623	5.6623	5.6623
	4	5.3689	5.6816	5.6620	5.6627	5.6626	5.6626
	5	5.3179	5.6928	5.6613	5.6627	5.6627	5.6627
HO3	1	3.6699	5.2710	5.4058	5.4743	5.4541	5.4746
	2	3.7283	5.4207	5.5821	5.6572	5.6342	5.6546
	3	3.6457	5.4231	5.5897	5.6554	5.6421	5.6623
	4	3.5733	5.4273	5.5899	5.6658	5.6424	5.6626
	5	3.5191	5.4313	5.5895	5.6658	5.6425	5.6627

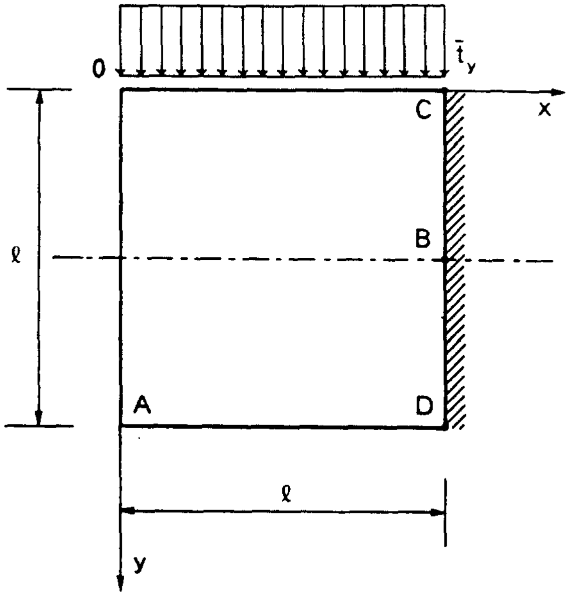

Fig. 3. Cantilever square plate clamped along edge CD.

(Fig. 3), is considered on purpose, as it seems to be a "bad" case. Indeed, a weak singularity should be expected in principle on stresses at corners C and D, between clamped and free edges[26]. This fact, as remarked by Benthem[27], renders any calculation procedure, if a regular stress field is assumed, very slow to converge to a result also in regions not close to the corners, where stresses are only moderate. Values taken by displacement u_y at point $A \equiv (0, l)$ and stresses s_x and s_{xy} at point $B \equiv (l, l/2)$ are reported in Table 5 for NS = 9 only, but they coincide up to the third figure at least with the results obtained by assuming NS = 7. An investigation on stress s_x at points C and D reveals no convergence toward a value when NS and NH are increased. Only the comparison between models ES5 with functions $X_j(x)$ given by relationships (9a) and (9b) is exposed, as the behavior of models ES3 and HS5 with respect to ES5 is the same as in the previous case. The author is unaware of exact solutions to be assumed as a reference for this case. Namely, Southwell's polynomial solution[28, Art. 413] is suitable only for slender, quasi-beam cantilever plates. For this reason, results obtained by discretizing the plate with a mesh of 9 × 9 equal, eight-node isoparametric

Table 4. Simply supported square plate uniformly loaded at the upper side (Fig. 2). Stress s_y at point C (multiplier: $-0.1 \cdot \bar{t}_y$)

		NS					
Models	NH	1	3	5	7	9	Filon
ES3	1	6.3662	5.4206	5.4121	5.4110	5.4108	5.4107
	2	4.2441	5.1864	5.1913	5.1924	5.1929	5.1932
	3	5.5174	5.2394	5.2014	5.2016	5.2018	5.2020
	4	4.6079	5.1754	5.1995	5.2011	5.2014	5.2016
ES5	1	5.3953	5.4106	5.4107	5.4107	5.4107	5.4107
	2	5.3610	5.1932	5.1932	5.1932	5.1932	5.1932
	3	5.1226	5.2021	5.2020	5.2020	5.2020	5.2020
	4	5.3481	5.3481	5.2016	5.2016	5.2016	5.2016
HS5	1	5.3953	5.4107	5.4107	5.4107	5.4107	5.4107
	2	5.3610	5.1917	5.1932	5.1932	5.1932	5.1932
	3	5.1226	5.2066	5.2019	5.2020	5.2020	5.2020
	4	5.3481	5.1947	5.2018	5.2016	5.2016	5.2016
HO3	1	5.2483	5.4264	5.4200	5.4299	5.4141	5.4107
	2	4.7556	5.1857	5.1837	5.1969	5.1894	5.1932
	3	4.8684	5.1864	5.1955	5.2078	5.1992	5.2020
	4	4.8269	5.1902	5.1951	5.2071	5.1987	5.2016

Table 5. Cantilever square plate uniformly loaded at the upper side (Fig. 3). Displacement u_y at point A and stresses s_x, s_{xy} at point B

Models	NH	$\hat{u}_y$	$\bar{u}_y$	$\hat{s}_x$	$\bar{s}_x$	$\hat{s}_{xy}$	$\bar{s}_{xy}$
ES5	1	2.6586	3.2081	.97022	.81444	11.461	8.7375
	2	2.7630	2.8614	1.2044	1.1228	10.374	9.7517
	3	2.8586	3.0493	1.1437	1.1422	9.9992	9.8658
	4	2.8731	2.9565	1.1180	1.1292	9.6359	9.6979
	5	2.9066	3.0195	1.1100	1.1226	9.5311	9.6368
	6	2.9113	2.9778	1.1064	1.1179	9.4123	9.5508
	7	2.9284	3.0091	1.1052	1.1152	9.3672	9.5035
	8	2.9303	2.9850	1.1044	1.1130	9.3119	9.4528
	9	2.9408	3.0039	1.1042	1.1116	9.2864	9.4188
	10	2.9417	2.9880	1.1040	1.1104	9.2549	9.3852
	14	2.9547	2.9899	1.1040	1.1077	9.1924	9.2993
	15	2.9586	2.9970	1.1040	1.1074	9.1837	9.2845
	19	2.9644	2.9949	1.1041	1.1063	9.1536	9.2381
	20	2.9643	2.9903	1.1041	1.1062	9.1474	9.2291
FE		2.9743		1.0383		9.0802	
Multiplier		$\bar{t}_y l/E$		$-0.1 \cdot \bar{t}_y$		$-0.1 \cdot \bar{t}_y$	

ˆ, with function (9a);
¯, with function (9b).

displacement finite elements[29] are also reported (code number: FE). It should be remarked that stresses s_x and s_{xy} in this solution concern the (elemental, Gauss) point of coordinates $x = 0.0125\ l$, $y = l/2$. Convergence appears to be slow, especially in comparison with the previous case, and the agreement of results obtained via functions (9a) and (9b) may be likely blurred also by this fact. More accurate results could be obtained by refining the strip subdivision close to upper and lower edges, as customary in the case of high-stress gradients, but an improvement of the model may be, in principle, pursued by incorporating in some way the singularity in the stress field description[27, 30].

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